Credit Solvency Capital Requirements.

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Abstract: This article constructs a recovery-based framework for computing the credit Solvency Capital Requirements of insurers under the constant position paradigm. Although this framework is most suited under the Solvency 2 regulation, it also provides concepts that can be useful under the Basel regulation. After a brief survey of the extant technology on rating transitions and default probabilities, the paper provides new results on risk premium adjustment factors. Then, three different procedures for reconstructing constant position market-consistent histories of credit portfolios from quoted Merryll Lynch indices are given. The reconstructed historical credit values are modeled via a mixed empirical-Generalized Pareto Distribution (GPD) dynamics and a detailed parameter estimation is performed. Several validations of the estimation are also provided. Finally, credit Solvency Capital Requirements are computed and an analysis of the results per rating class is given.

Keywords: Credit Spread, Risk Premium Adjustment Factor, Solvency Capital Requirement, General Pareto Distribution, Market Consistency, Rating Transition, Credit Benchmarking, Constant Position.

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Introduction

According to the Solvency 2 regulation, insurers need to be able to assess the capital needs that cover the risk of annual losses due to credit risk. The applications can be as well for Own Risk and Solvency Assessments as for computing internal model Solvency Capital Requirements (SCRs). Being able to measure credit risk is also an important precondition for the asset management of insurers. This paper introduces an original and time-efficient solution to the problem of linking credit portfolios to credit market indices and of computing credit SCRs. The solution uses a pricing scheme that decomposes credit spreads into default probabilities and losses given default, so that takes into account recovery risk.

It is usually impossible to directly build an aggregate index that perfectly reflects the risk profile of the credit portfolio of any given investor. Indeed, the recovery rates of the assets constituting a credit market index are usually quite homogeneous by construction, whereas investors build up credit portfolios by selecting assets with varying recovery rates. For instance, investors can select bonds of low rating and high recovery and bonds of high rating and low recovery, and such a strategy cannot be directly replicated using existing market credit indices. An additional difficulty lies in the limited ability of the investor or the market to determine the recovery rates of credit instruments. However, to quantify spread risk, we need to start from credit market indices. In this paper, we use past available index data to construct pseudo-indices that mimic target credit portfolios in all aspects except recovery risk. These pseudo-indices constitute an important step toward the reconstruction of market-consistent credit observations, where a final adjustment for recovery risk is made. Using a one-year GPD distribution to model the reconstructed credit observations allows us to achieve a quantization of spread risk and to compute SCRs and similar indicators.

Crouhy, Galai and Mark (2000) and Bruyère *et al* (2006) provide broad discussions on the credit models used by banks. Credit risk can be tackled using either a structural approach or a reduced form approach, the latter of which can be intensity-based or not. The structural approach is based on an economic model of the corporation. See for instance Leland (1994), Longstaff and Schwartz (1995) or Collin-Dufresne and Goldstein (2001). The reduced form approach directly measures default and rating transitions. See for instance Elliott, Jeanblanc and Yor (2000), Bielecki and Rutkowski (2002), or Schönbucher (2003). Our paper is based on the reduced form approach constructed in Jarrow, Lando and Turnbull (1997) and in Israel, Rosenthal and Wei (2001). For general perspectives on recovery rates, see Bruche and González-Aguado (2010), Schneider, Sögner and Veža (2010), or Schläfer and Uhrig-Homburg (2014). Cohen and Costanzino (2015a, 2015b) provide a framework for modeling stochastic recovery rates. For a general presentation of the Solvency 2 and Basle regulations, we refer the reader to Gatzert and Wesker (2011). Gatzert and Martin (2012) discuss the standard approach versus internal models. Among the references that concentrate on tail risk, we can cite Meine, Supper and Weiss (2016). Our Generalized Pareto Distribution approach uses the contribution of Hill (1975) and Hosking and Wallis (1987).

The paper is organized as follows. In a first section, we introduce the main elements of the pricing approach and we also provide new results on risk-premium adjustment factors. In a second section, the paper develops a set of methods that allow us to extract relevant credit experience. We construct, based on Merrill Lynch credit indices, a primary pseudo-index that is representative of all the elements - except recovery - of the risk profile of a given credit portfolio. Then, we extract values of the risk premium adjustment factor from the pseudo-index. Then, we recombine this information and the recovery rates of the constituents of the credit portfolio to reconstruct a synthetic market-consistent history of the credit portfolio. In the third section of the paper, a mixed empirical-Generalized Pareto Distribution process is used to model the one year risk of the credit portfolio. An ample discussion and validation of the estimation of the model parameters is provided. In the fourth and final section, an illustration is conducted where gross Solvency Capital Requirements are computed and discussed with regard to the rating class and recovery rate.

1 Credit Modeling

We present here the credit risk framework used in this article. For this purpose, we follow the lines of Jarrow, Lando, and Turnbull (1997).

1.1 Historical Rating Transitions

We start by describing rating transitions in the real world. Denote by κ_t the rating of an issuer at time t. Then, the probability of a transition from the rating κ_t to the rating $\kappa_{t'}$ between times t and t' is written as

$$P(\kappa_{t'} = h | \kappa_t = k) = p_{h,k}(t, t').$$

Assume that there are K viable states. Next, define the default time τ as the first time when κ_t enters into an additional state $K+1$, which is the default state. We have:

$$\tau = \inf\{t \geq 0 \mid \kappa_t = K+1\},$$

where $K+1$ is an absorbing state.

The set of rating transition probabilities constitutes the transition matrix over $K+1$ possible states, denoted as

$$T^P(t, t') = \begin{bmatrix} p_{1,1}(t,t') & \dots & p_{1,K}(t,t') & p_{1,K+1}(t,t') \\ \vdots & & \vdots & \vdots \\ p_{K,1}(t,t') & \dots & p_{K,K}(t,t') & p_{K,K+1}(t,t') \\ 0 & \dots & 0 & 1 \end{bmatrix},$$

where the superscript P indicates that the matrix is considered in the real-world.

The sum of the coefficients of T^P along each line is equal to 1. The last line of T^P is comprised of zeros and of a final coefficient equal to 1: this corresponds to the fact that $K+1$ is an absorbing state.

We typically assume that κ is a time-homogeneous Markov chain in discrete time, so that $T^P(m, n) = T^P(m+l, n+l)$ when m, n, and l are integers. Then, note that we have the following master equation between times 0 and 2:

$$p_{h,k}(0, 2) = \sum_{i=1}^{K+1} p_{h,i}(0, 1) \times p_{i,k}(1, 2),$$

or, in matrix terms,

$$T^P(0,2) = T^P(0,1)\, T^P(1,2),$$

or, assuming time-homogeneity,

$$T^P(0,2) = T^P(0,1)^2.$$

By extension, we have, for all N:

$$T^P(0,N) = T^P(0,1)^N,$$

which gives

$$T^P(0,t) = T^P\left(0, \frac{t}{N}\right)^N$$

or

$$T^P(0,t) = \left(1 + \frac{N\left(T^P\left(0, \frac{t}{N}\right) - 1\right)}{N}\right)^N.$$

Asymptotically, we write:

$$T^P(0,t) = e^{\Lambda^P t} = \sum_{i=0}^{+\infty} \frac{\left(t\Lambda^P\right)^i}{i!},$$

where Λ^P is called the generating matrix or generator. The sum of coefficients along each line of this matrix is null and all the coefficients of its last line are null.

Assume that the generator is diagonalizable, so that

$$\Lambda^P = XDX^{-1},$$

where D is a diagonal matrix and X is an invertible matrix whose columns are the eigenvectors of Λ^P. Then, for all i,

$$\left(\Lambda^P\right)^i = XD^iX^{-1}$$

yields

$$T^P(0,t) = \sum_{i=0}^{+\infty} X \frac{(tD)^i}{i!} X^{-1}$$

and

$$T^P(0,t) = Xe^{tD}X^{-1}.$$

which gives a direct way of computing T^P knowing Λ^P. The converse operation can be achieved using the formulas in Jarrow, Lando, and Turnbull (1997)):

$$\begin{cases} \Lambda^P_{k,k} = \ln\left(T^P_{k,k}(0,1)\right) \\ \\ \Lambda^P_{h,k} = \ln\left(T^P_{k,k}(0,1)\right) \frac{T^P_{h,k}(0,1)}{T^P_{k,k}(0,1)-1} \end{cases},$$

or the improved formula of Israel, Rosenthal, and Wei (2001):

$$\Lambda^P = -\frac{1}{t}\sum_{i=1}^{+\infty} \frac{\left(I - T^P(0,t)\right)^i}{i}.$$

1.2 Risk-Neutral Rating Transitions and Risk Premium Adjustment Factors

Let us now come to the study of risk-neutral rating transitions. The probabilities of these transitions are denoted as follows:

$$Q(\kappa_{t'} = h | \kappa_t = k) = q_{h,k}(t,t'),$$

and we introduce, similar to the historical transition matrix, the risk-neutral transition matrix:

$$T^Q(t,t') = \begin{bmatrix} q_{1,1}(t,t') & \dots & q_{1,K}(t,t') & q_{1,K+1}(t,t') \\ \vdots & & \vdots & \vdots \\ q_{K,1}(t,t') & \dots & q_{K,K}(t,t') & q_{K,K+1}(t,t') \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

Whereas it is possible to use a constant generating matrix in the historical case, it is difficult to maintain consistency between the historical and the risk-neutral universe and to have a constant risk-neutral generating matrix. Therefore, we assume in full generality that

$$T^Q(0,t) = e^{\int_0^t \Lambda^Q(s)ds},$$

where this expression has to be understood in the symbolic sense as the solution of the following matrix differential equation:

$$\frac{dT^Q(0,t)}{dt} = T^Q(0,t)\Lambda^Q(t).$$

Several ways to relate historical and risk-neutral transition probabilities exist. In each approach, a set of risk premium adjustement factors is introduced. For instance, we can write

$$\begin{cases} q_{h,k}(t,t+1) = \psi_{h,k}(t)\ p_{h,k}(t,t+1) \quad h \neq k \\ q_{k,k}(t,t+1) = 1 - \sum\limits_{h \neq k} q_{h,k}(t,t+1), \end{cases}$$

where the factors $\psi_{h,k}$ are introduced to relate all the possible historical and risk-neutral transitions. In a simplified approach, the factors only depend on the initial rating, yielding the series of equations:

$$\begin{cases} q_{h,k}(t,t+1) = \psi_k(t)\ p_{h,k}(t,t+1) \quad h \neq k \\ q_{k,k}(t,t+1) = 1 - \sum\limits_{h \neq k} q_{h,k}(t,t+1) = 1 - \psi_k(t) \sum\limits_{h \neq k} p_{h,k}(t,t+1) \end{cases},$$

or, equivalently,

$$\begin{cases} q_{h,k}(t,t+1) = \psi_k(t)\ p_{h,k}(t,t+1) \quad h \neq k \\ q_{k,k}(t,t+1) = 1 - \psi_k(t)(1 - p_{k,k}(t,t+1)) \end{cases},$$

which can be simplified as follows:

$$T^Q(t,t+1) = I + \Psi(t)\left(T^P(t,t+1) - I\right),$$

where $\Psi(t)$ is the diagonal matrix with elements $(\psi_1(t), \ldots, \psi_K(t), 1)$.

In a distinct approach, we introduce the risk premium adjustment factor Π that satisfies

$$\Lambda^Q(t) = \Pi(t)\Lambda^P$$

where $\Pi(t)$ is the diagonal matrix with elements $(\pi_1(t), \ldots, \pi_K(t), 1)$. From this expression, we obtain:

$$T^Q(0,t) = e^{\int_0^t \Pi(s)\Lambda^P ds}.$$

A nearly identical approach consists in writing the following approximation:

$$T^Q(0,t) \approx e^{t\ \Pi\ \Lambda^P}$$

or in applying

$$T^Q(t,t+1) \approx e^{\Pi\ \Lambda^P}$$

recursively over discrete time steps. This is the approach retained in the next sections of this article.

Finally, the computation of default probabilities provides an illustration of a recombination of the above two approaches:

$$\begin{aligned} Q(\tau < T_N | \kappa_0 = k) = q_{k,K+1}(0,T_N) &= \left[e^{\int_0^{T_N} \Pi(s)\Lambda^P ds}\right]_{k,K+1} \\ &= \psi_{k,K+1}(0,T_N)\ p_{k,K+1}(0,T_N). \end{aligned} \tag{1}$$

For simplicity of notation, we drop the conditioning in the risk-neutral probability in the next computations.

1.3 Portfolio Implied Risk Premium Adjustment Factor

We now show how it is possible to extract relations between risk premium adjustment factors. We consider a portfolio of bonds that pertain to K rating classes. We assume that there are M_k bonds in each rating class k, so that the total number of bonds in the portfolio is $\sum_{k=1}^K M_k$. Although it is possible to interpret M_k as a number of maturity classes, we do not make this specification here and we only interpret M_k as a number of bonds of varying characteristics but common rating k. The total value of the portfolio can be expressed as

$$V = \sum_{k=1}^{K} \sum_{j=1}^{M_k} V^{j,k},$$

where $V^{j,k}$ is the value of the j^{th} bond of rating k. Denoting as $N^{j,k}$ the number of cash-flows of this bond and as T_i a cash-flow payment time, we have:

$$V^{j,k} = \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0,T_i) \left(R^{j,k} + (1-R^{j,k})(1 - Q(\tau^{j,k} < T_i))\right),$$

where $C_i^{j,k}$ is a coupon or principal payment at time T_i, $R^{j,k}$ is the recovery rate, and $\tau^{j,k}$ is the default time of the j^{th} bond of rating k.

Assuming that all bonds pay coupons over the same discrete set of dates, but allowing for varying numbers of coupons, we have:

$$V = \sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \left(R^{j,k} + (1 - R^{j,k})(1 - Q(\tau^{j,k} < T_i)) \right),$$

which can be rewritten as

$$V = \sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \left(R^{j,k} + (1 - R^{j,k})(1 - \psi_{k,K+1}(0, T_i) P(\tau^{j,k} < T_i)) \right),$$

The value of the portfolio can also be expressed as the sum of two components:

$$\begin{aligned} V = & \sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \\ & - \sum_{k=1}^{K} \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \psi_{k,K+1}(0, T_i) p_{k,K+1}(0, T_i), \end{aligned} \tag{2}$$

where the first component is the value of a portfolio of otherwise equivalent credit-risk-free bonds.

Note that V^k, which is the value of the subportfolio of bonds of rating k, can be written as follows:

$$\begin{aligned} V^k = & \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} \; P(0, T_i) \\ & - \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} \; P(0, T_i) \; \psi_{k,K+1}(0, T_i) \; p_{k,K+1}(0, T_i). \end{aligned} \tag{3}$$

Determination of $(\pi_1, \ldots, \pi_k)$

We assume that $\Pi(t)$ is time-invariant and is a diagonal matrix with elements $(\pi_1, \ldots, \pi_K, 1)$. From Eq. (1), we can write

$$\left[e^{\mathrm{diag}(\pi_1, \ldots, \pi_K, 1) \Lambda^P t} \right]_{k,K+1} = \psi_{k,K+1}(0, t) \; p_{k,K+1}(0, t),$$

so that

$$\begin{aligned} V^k = & \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \\ & - \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \left[e^{\mathrm{diag}(\pi_1, \ldots, \pi_K, 1) \Lambda^P T_i} \right]_{k,K+1}. \end{aligned} \tag{4}$$

Therefore, denoting by $\hat{V}^k$ the quoted value of the subportfolio of bonds of rating k, the parameters $(\pi_1, \ldots, \pi_K)$ can be determined by solving the system of K equations:

$$\left\{ \forall k = 1 : K \quad \hat{V}^k = V^k \right\}. \tag{5}$$

Case of a unique coefficient π

If we introduce the constraint of a unique coefficient $\pi = \pi_1 = \cdots = \pi_K$, then, it is not anymore possible to simultaneously solve the K equations (5). The calibration can be performed by minimizing a distance criterion. For instance, it is possible to solve:

$$\min_{\pi} \sum_{k=1}^{K} \Delta_k^2, \tag{6}$$

where

$$\begin{aligned}\Delta_k = \hat{V}^k &- \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0,T_i) \\ &- \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0,T_i) \left[e^{\mathrm{diag}(\pi,\ldots,\pi,1)\Lambda^P T_i} \right]_{k,K+1}.\end{aligned} \tag{7}$$

The solution π to Eq. (6) does not allow us to exactly reproduce prices but only to minimize the quadratic distance between model and empirical bond prices.

Similarly, it is possible to solve the following program:

$$\min_{\pi} \sum_{k=1}^{K} \sum_{j=1}^{M_k} \Delta_{j,k}^2,$$

where $\Delta_{j,k}$ is defined per rating and per maturity class.

The calibration can also be performed by setting $V = \hat{V}$ where $\hat{V}$ is the quoted value of the portfolio. For simplicity, we retain this latter approach in the remainder of the paper when π is assumed unique.

Computation of a unique equivalent coefficient ψ

Using Eq. (2), we search for the unique risk premium adjustment factor ψ that does not depend on the rating or the maturity of bonds and that is equivalent to the set of factors $\psi_{k,K+1}(0,T_i)$. We write:

$$\begin{aligned}V = &\sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0,T_i) \\ &- \sum_{k=1}^{K} \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0,T_i)\ \psi\ p_{k,K+1}(0,T_i),\end{aligned} \tag{8}$$

which yields

$$\psi = \frac{\sum_{k=1}^{K} \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k}\ P(0,T_i)\ \psi_{k,K+1}(0,T_i)\ p_{k,K+1}(0,T_i)}{\sum_{k=1}^{K} \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k}\ P(0,T_i)\ p_{k,K+1}(0,T_i)},$$

or, in a more compact form,

$$\psi = \frac{\sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} w_i^{j,k} \; \psi_{k,K+1}(0, T_i)}{\sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} w_i^{j,k}}, \tag{9}$$

where

$$w_i^{j,k} = (1 - R^{j,k}) \; C_i^{j,k} \; P(0, T_i) \; P(\tau^{j,k} < T_i).$$

Note that Eq. (9) can be simplified into

$$\psi = w \sum_{k=1}^{K} \sum_{i=1}^{N^{j,k}} \psi_{k,K+1}(0, T_i),$$

provided the weights $w_i^{j,k}$ are sufficiently regular. This is only possible under specific investment conditions: the portfolio manager should select bonds whose recovery rate evolves in the same direction as their probability of default.

2 Credit Benchmarking

Historical bond portfolio values cannot be directly used for computing Solvency Capital Requirements. Indeed, the composition of an insurance company's bond portfolio may substantially change with the passage of time. In this section, we examine how it is possible to reconstruct historical databases of benchmarked credit portfolio values under the constant position paradigm, which assumes that past portfolio weights have not been modified and are identical to current portfolio weights. This amounts to reconstructing the history of a virtual portfolio that has never been rebalanced and whose weights are the current portfolio weights. The historical values obtained in this section will be subsequently used for computing credit SCRs.

We introduce three approaches for constructing historical portfolio values. The first approach uses a global index, the second approach decomposes the credit portfolio into a sub-portfolio of bonds with full recovery and a sub-portfolio of remaining bonds, and the third approach, which is the most precise of the three, reconstructs historical values for subportfolios of identical notation before producing an aggregate indicator.

In each of these approaches, the determination of the risk premium adjustment factor Π is a key step. In the first two approaches, a unique constant factor π is assumed, while in the third approach we use a vector $(\pi_1, \ldots, \pi_k)$ of factors that are constant by notation.

Examples of credit indexes that can be used for benchmarking are the Merrill Lynch monthly indexes that reflect the performance of corporate bonds of the Euro zone. These indexes are comprised of sub-indexes computed by rating and by maturity. In order to neutralize credit-risk-free interest rate movements, we subtract from these corporate indexes another Merrill Lynch index that accounts for Government bond movements. Let us now describe our three credit benchmarking approaches.

All the computations of this section use the valuation formula:

$$
\begin{aligned}
V = &\sum_{k=1}^{K}\sum_{j=1}^{M_k}\sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0,T_i) \\
&- \sum_{k=1}^{K}\sum_{j=1}^{M_k}(1-R^{j,k})\sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0,T_i)\left[e^{\mathrm{diag}(\pi_1,\ldots,\pi_K,1)\Lambda^P T_i}\right]_{k,K+1}, \qquad (10)
\end{aligned}
$$

which is obtained by summing up the contributions in Eq. (4). In approaches 1 and 2, $\mathrm{diag}(\pi_1,\ldots,\pi_K,1)$ is simply $\mathrm{diag}(\pi,\ldots,\pi,1)$.

2.1 First Approach

We present a first approach for recomputing the historical values of a credit portfolio in the constant position paradigm. As an intermediate step, we construct a credit pseudo-portfolio, whose composition by rating and maturity is similar to that of the credit portfolio but whose average recovery rate differs. This approach heavily relies on Eq. (10), where a unique coefficient π is assumed and where the historical values of this coefficient are recovered from the historical values of the credit index.

The first approach consists in eight steps. The first sixth steps produce a history of values $V^{P'}(t)$ for a pseudo-portfolio P' that mimicks the portfolio P of the investor using on the past values and recovery rates of a market index I. The last two steps use the pseudo-portfolio to reconstruct the past values $V^P(t)$ of the credit portfolio. Specifically, in the last step of the algorithm, the effective recovery rates of the portfolio are incorporated in lieu of the recovery rates of the index. The detailed algorithm is as follows.

In a first step, the monthly returns of several credit sub-indexes are obtained from a database. For instance, $r^I_{\kappa,m}(t)$ gives the return at time t of the credit sub-index of rating κ and maturity m. We assume that there are K possible ratings (excluding the bankruptcy state) and M possible maturity ranges.

In a second step, we obtain the current value of the credit portfolio for each rating and maturity layer by observing the current quotes of the bonds in the portfolio. For example, $V^P_{\kappa,m}(0)$ gives the value at time 0 of the credit sub-portfolio of rating κ and maturity m. Using this information, we can compute the weights at time 0 of the sub-portfolios within the total credit portfolio:

$$
w^P_{\kappa,m}(0) = \frac{V^P_{\kappa,m}(0)}{\sum\limits_{\kappa,m} V^P_{\kappa,m}(0)}.
$$

The next step consists in computing $\pi^P(0)$, which is the risk premium adjustment factor at time 0 for the credit portfolio. This quantity is obtained by solving Eq. (10), where $V = V^P(0)$ is the total credit portfolio value.

For the fourth step, we multiply the monthly return (at any time $t \leq 0$) of each sub-index by the weight invested at time 0 in the corresponding sub-portfolio of identical rating and maturity. This is precisely the "constant position" approach, which can be written as follows:

$$
r^P(t) = \sum_{\kappa,m} r^I_{\kappa,m}(t)\; w^P_{\kappa,m}(0) \quad \forall t \leq 0,
$$

where r^P is a reconstruction of the past credit portfolio returns.

In a fifth step, we start by computing the average recovery rate R^I of the index and we assume this quantity is constant. The computation is performed using for instance:

$$R^I = \frac{1}{KM} \sum_{\kappa,m} R^I_{\kappa,m},$$

but alternative averaging methods are possible. Then, we compute the initial value $V'^P(0)$ of the pseudo-portfolio using Eq. (10). This evaluation is conducted using the average recovery rate R^I of the index instead of the recovery rates $R^P_{\kappa,m}$ of the credit portfolio. In this evaluation, we also use the value $\pi^P(0)$ computed above.

The sixth step consists in starting from the value $V'^P(0)$ and in extrapolating to other values $V'^P(t)$ by discounting with the values of r^P. We have:

$$V'^P(t) = \frac{V'^P(0)}{\prod_{t_i}(1 + r^P(t_i))},$$

where the historical values that are computed are for the credit pseudo-portfolio, so for a credit portfolio whose recovery rate is that of the index (but all other determinants are kept unchanged).

In the penultimate step, we use the pseudo-portfolio to compute the historical values $\pi^P(t)$. The past values of the risk premium adjustment factor are the solutions to Eq. (10) when the recovery rate is R^I and the values $V'^P(t)$ are retained.

Finally, in the last step, we obtain the historical values $V^P(t)$ of the credit portfolio using Eq. (10) with the recovery rates $R^P_{\kappa,m}$ of every subclass of the portfolio and with the risk premium adjustment factor $\pi^P(t)$.

2.2 Second Approach

The second approach keeps the assumption of a unique factor π but extends the first approach by distinguishing assets with a recovery rate of 100% from other assets. The necessity to operate this distinction comes from the empirical observation that some defaults do not induce costs or induce minor costs. This is the case when a company with valuable tangible assets defaults because of a lack of cash in the short run. Indeed, after such a default (administered under the Chapter 11 of the US bankruptcy code, or under similar codes in other countries), the company can start afresh and bondholders can ultimately recover the nominal value of their bonds. In the second approach, we distinguish such companies from other companies who incur substantial losses in case of a default.

Most of the algorithm that computes the past values $V^P(t)$ of the credit portfolio is unchanged: only the second and last steps of the algorithm need to be modified in order to go from approach 1 to approach 2.

In the new second step of the algorithm, we compute from public data the market value at time 0 of the bonds whose recovery rate is different from 100% and whose rating and maturity are κ and m, respectively. The value of these bonds is denoted $\tilde{V}^P_{\kappa,m}(0)$, by class of rating and maturity. It is also possible to compute the weight of each class as follows:

$$\tilde{w}^P_{\kappa,m}(0) = \frac{\tilde{V}^P_{\kappa,m}(0)}{\sum\limits_{\kappa,m} \tilde{V}^P_{\kappa,m}(0)}.$$

Further, the value of the bonds that have a full recovery rate is computed as an aggregate over all ratings and maturities; it is denoted $\hat{V}^P(0)$.

The next steps of the algorithm are unchanged and yield a value $\tilde{V}'^P(0)$ for the credit pseudo-portfolio. This value is computed from Eq. (10), where the average recovery rate R^I of the index is used. As in the first version of the algorithm, the historical values $\tilde{V}'^P(t)$ of the pseudo-portfolio are then deduced. The historical values $\pi^P(t)$ are also computed as in the first algorithm, based on the values $\tilde{V}'^P(t)$.

In the modified last step of the algorithm, we first compute the historical values of the portfolio of bonds with partial recovery, namely $\tilde{V}^P(t)$. Then, for the bonds with full recovery, and because we have canceled risk-free rate effects, we write:

$$\forall t \quad \hat{V}^P(t) = \hat{V}^P(0).$$

Next, the aggregate historical value of the credit portfolio is estimated as follows:

$$V^P(t) = \tilde{V}^P(t) + \hat{V}^P(t).$$

2.3 Third Approach

The third approach extends the first one by considering a distinct risk premium adjustment factor for each rating class. Nearly all the steps of the algorithm of approach 3 are identical to the steps of approach 1 and use Eq. (10). Only step 7 differs: it now solves the system (5) for determining $\pi_{1,\dots,k}$. Note from Eq. (4) that the value of each subportfolio k depends on all the values $\pi_{1,\dots,k}$.

2.4 Illustration

Let us now compare the three approaches introduced in this section. For this illustration, we constitute a portfolio P comprised of 16 iso-weighted layers of corporate bonds. All bonds pertain to the investment grade class. Four bonds have been selected in each of the four investment grade ratings. See Table 6 in the appendix for the full dataset.

In Figure 1, we show the reconstructed bond portfolio historical values for each of the three approaches. Interestingly, approach 2 reconstructs historical data that is very close to the data reconstructed by approach 3 (which is by construction the most accurate of the three approaches). Therefore, separating bonds with a full recovery from other types of bonds can be sufficient to reach sufficient pricing reliability without having to postulate several values of π. Finally note that approach 1 severely underprices bond portfolios. The next section studies the estimation of stochastic models to the data reconstructed here.

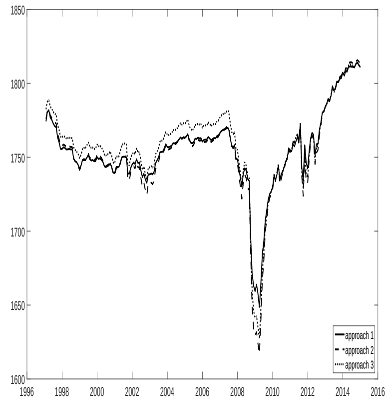

Figure 1 – Credit Portfolio History, Approaches 1, 2, and 3

3 Dynamic Credit Portfolios

First, we propose a model for representing the dynamics of the credit portfolio V. Then, we estimate the parameters of this model and we provide several validation methods for this estimation.

3.1 Model

We first address the following question: should V be modeled using Cox-Ingersoll-Ross dynamics? We show in Figure 2 the autocorrelations of the daily returns and squared returns of the credit portfolio. For computing these graphs, we use the historical values of V computed in the previous section using the third approach. The autocorrelations shown in the figure are important even after several days. We deduce from this observation that a good model for representing the dynamics of daily movements of V should incorporate some dependence of increments. Therefore a CIR process could be an appropriate candidate for this purpose.

However, the knowledge of the daily dynamics of V is not required for the applications considered in this article. Indeed, we need to work over longer horizons because we are computing Solvency Capital Requirements on an annual basis. Let us examine how the dynamics change when we increase the time horizon. We now show in Figure 3 the autocorrelations of the monthly returns and squared returns of the credit portfolio (using again the historical data reconstructed with the third approach of the previous section). From these graphs, it appears that the autocorrelations quickly remain within the confidence bars of the no autocorrelation hypothesis. This means that the dynamics of V present little or no autocorrelation at a sufficiently low frequency. Therefore, an appropriate continuous-time process used for modeling these dynamics does not necessarily need to incorporate mean-reversion.

Based on the previous analysis, we model the dynamics of V using a process with independent increments. The marginal distributions of this process

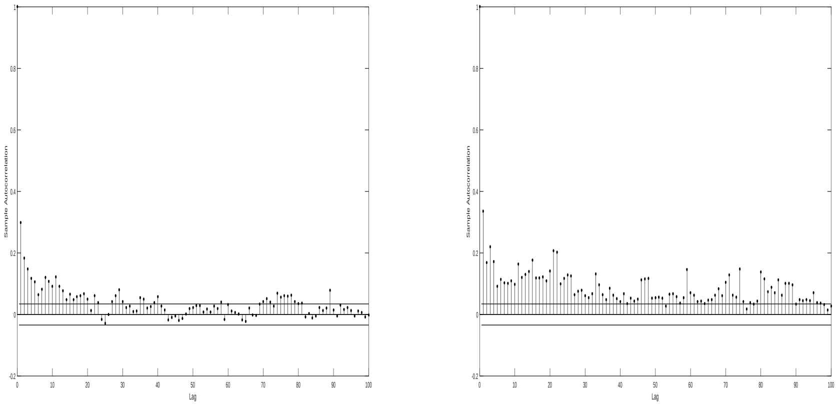

Figure 2 – Autocorrelations of Daily Returns and Squared Returns of V

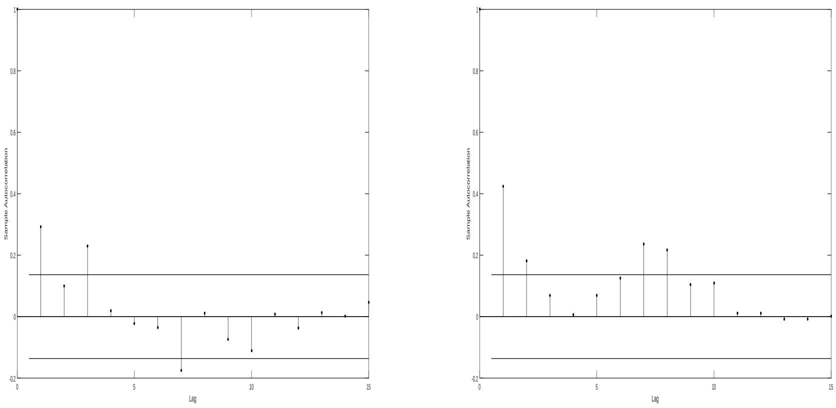

Figure 3 – Autocorrelations of Monthly Returns and Squared Returns of V

are specified as a mixed GPD-empirical distribution. The justifications for constructing such a model are as follows. Because we have enough data for small and medium size fluctuations, we choose to keep the empirical distribution in this range of fluctuations. However, we do not have enough large empirical points to construct a viable distribution in the tail. We model the extreme value part of the distribution with a Generalized Pareto Distribution (GPD). When applied to positive extreme values, this latter distribution can be written as follows:

$$G_{u,\xi,\sigma}(x) = 1 - \left(1 + \frac{\xi}{\sigma}(x-u)\right)^{-\frac{1}{\xi}} \tag{11}$$

for all $x \geq u$ and assuming $\xi \neq 0$.

A similar expression is available for negative extreme values. Also note that the probability density function is

$$g_{u,\xi,\sigma}(x) = \frac{1}{\sigma}\left(1 + \frac{\xi}{\sigma}(x-u)\right)^{-\frac{1}{\xi}-1}$$

for all $x > u$.

3.2 Estimation

We model V using the Generalized Pareto Distribution and develop a method that complements the approach shown in Le Courtois and Walter (2014). This method integrates in a general scheme the contributions of Hill (1975) and Hosking and Wallis (1987). However, these contributions are not used in a classic way. Before presenting the general scheme, we recall the approaches of Hill and Hosking and Wallis.

The Hill method is traditionally used for the estimation of a strictly positive shape parameter ($\xi > 0$). Starting from a sample $x_i, i = 1, \cdots, n$ of independent identically distributed observations of identical sign, ranked observations are constructed. Specifically, $x_{k,n}$ is the k-th observation in decreasing order. The Hill method then consists in representing graphically the values of

$$\hat{\alpha}_{k,n} = \left(\frac{1}{k}\sum_{j=1}^{k} \ln x_{j,n} - \ln x_{k,n}\right)^{-1}$$

with respect to k where $k = 1, \cdots, n$. The coefficients $\hat{\alpha}_{k,n}$ describe the mean behavior of exceedances beyond the thresholds $x_{k,n}$. Under the GPD assumption, $\hat{\alpha}_{k,n} \to 1/\xi$ when $n \to +\infty$, $k = k(n) \to +\infty$, and $k/n \to 0$.

Hosking and Wallis have shown that using moments can be more efficient for the estimation of β and ξ than using other methods such as the maximum likelihood one. They provide the following estimators:

$$\hat{\xi} = 2 - \frac{w_0}{w_0 - 2\ w_1}$$

and

$$\hat{\beta} = \frac{2\ w_0\ w_1}{w_0 - 2\ w_1},$$

with

$$w_0 = E(X)$$

and

$$w_1 = E\left(X(1 - G(X))\right),$$

where X is a GPD random variable of distribution function G.

Our algorithm uses the Hill method as a tool for obtaining the threshold u instead of the coefficient ξ. The Hosking and Wallis method is used both in an intermediate step of the algorithm and in the final step for the computation of ξ and σ. The algorithm can be described as follows.

In a first step, we construct candidate thresholds that depend on an arbitrary level of precision h:

$$\hat{u}(i) = x_{n,n} + i\,h,$$

where

$$0 \leq i \leq \frac{x_{1,n} - x_{n,n}}{h}.$$

In a second step, we compute the estimators $\hat{\sigma}(\hat{u}(i))$ and $\hat{\xi}(\hat{u}(i))$ for the generalized Pareto distributions associated with the thresholds $\hat{u}(i)$ and using the method of Hosking and Wallis (1987).

In the next step, we construct translated ordered observations:

$$\tilde{x}_{k,n} = x_{k,n} - \hat{u}(i) + \frac{\hat{\sigma}(\hat{u}(i))}{\hat{\xi}(\hat{u}(i))},$$

to recover simple Pareto-like distributions for which Hill coefficients are known to be more stable.

In a fourth step, we compute the Hill coefficients for the values of k for which $x_{k,n} > \hat{u}(i)$. So, for i given, we compute:

$$\hat{\alpha}_{i,k,n} = \left(\frac{1}{k}\sum_{j=1}^{k} \ln \tilde{x}_{j,n} - \ln \tilde{x}_{k,n}\right)^{-1},$$

for all k such that $1 \leq k \leq m(i)$ where $m(i)$ is the first integer satisfying $x_{m(i)+1,n} \leq \hat{u}(i)$. This step is illustrated in Figure 4 that represents the Hill coefficient as a function of k for both the negative and positive extremes of the historical bond portfolio data reconstructed using the third approach of the previous section.

Then, for each i, we compute a coefficient of dispersion S_i measuring the stability of the Hill plot comprised of the points $(k, \hat{\alpha}_{i,k,n})$. This coefficient is defined as follows:

$$S_i = \frac{1}{m(i)\;\hat{\xi}(\hat{u}(i))} \sum_{j=1}^{m(i)} \left| \hat{\alpha}_{i,j,n} - \frac{1}{m(i)} \sum_{k=1}^{m(i)} \hat{\alpha}_{i,k,n} \right|.$$

In a sixth step, the optimal threshold is computed as $\hat{u}(i_{\text{opt}}) = r_{n,n} + i_{\text{opt}}\;h$ where i_{opt} is the integer that minimizes the coefficient of dispersion:

$$S_{i_{\text{opt}}} = \inf\left(S_i \middle|\; 0 \leq i \leq \frac{x_{1,n} - x_{n,n}}{h}\right).$$

The final step consists in extracting the Hosking and Wallis estimators $\hat{\sigma}(\hat{u}(i_{\text{opt}}))$ and $\hat{\xi}(\hat{u}(i_{\text{opt}}))$.

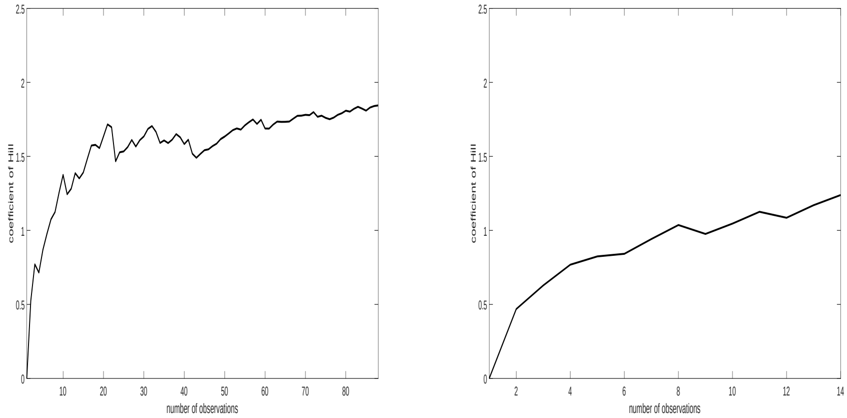

Figure 4 – Hill graphs for negative and positive extremes.

	$\hat{u}^n$	$\hat{\sigma}^n$	$\hat{\xi}^n$	$\hat{u}^p$	$\hat{\sigma}^p$	$\hat{\xi}^p$
Approach 1	0.00319	0.00132	0.61115	0.00411	0.00126	0.48838
Approach 2	0.00016	0.00134	0.49649	0.00561	0.00214	0.45648
Approach 3	0.00013	0.00129	0.48936	0.00483	0.00176	0.46078

Table 1 – GPD parameters.

To conclude, we apply this algorithm to estimate the GPD parameters of our database of historical bond portfolio values constructed using approaches 1, 2, or 3. We estimate the parameters $\hat{u}^n$, $\hat{\sigma}^n$ and $\hat{\xi}^n$ for the negative extremes and $\hat{u}^p$, $\hat{\sigma}^p$ and $\hat{\xi}^p$ for the positive extremes. The results are shown in Table 1.

3.3 Validation

We use four methods to confirm the validity of the parameters estimated in the previous section.

Maximum likelihood

Assuming the thresholds $\hat{u}^n$ and $\hat{u}^p$ are as obtained in Table 1, we compute the maximum likelihood estimates $\hat{\sigma}^n$, $\hat{\xi}^n$, $\hat{\sigma}^p$, and $\hat{\xi}^p$. To do so, we maximize the log-likelihood function of ξ and σ:

$$\ln(\mathcal{L}) = \sum_{i=1}^{n} \ln\left(g_{u,\xi,\sigma}(x_i)\right) = -n\ln(\sigma) - \left(\frac{1}{\xi}+1\right)\sum_{i=1}^{n}\ln\left(1+\frac{\xi}{\sigma}(x_i-u)\right)$$

for both negative and positive extremes.

	$\hat{\sigma}^n$	$\hat{\xi}^n$	$\hat{\sigma}^p$	$\hat{\xi}^p$
Approach 1	0.00121	0.84394	0.00136	0.55780
Approach 2	0.00129	0.58651	0.00296	0.28084
Approach 3	0.00126	0.56271	0.00215	0.39885

Table 2 – Maximum likelihood estimates.

While the maximum likelihood and the extended Hill estimates classically do not coincide, the results of this experiment, shown in Table 2, can be used to confirm the order of magnitude of the estimates $\hat{\sigma}^n$, $\hat{\xi}^n$, $\hat{\sigma}^p$, and $\hat{\xi}^p$ shown in Table 1.

Lorenz curves and Gini coefficients

We construct Lorenz curves as follows. We rank excess returns beyond u in decreasing order. The abscissa of the curves represents the proportions of such observations. The coordinate represents the accumulation of excess returns (the sum of all excess returns is normalized to 1 for convenience).

Figure 5 shows the Lorenz curves graphed for both the negative and positive excess returns of the historical bond portfolio data reconstructed using the third approach of the previous section. The plain lines represent the Lorenz curves obtained for the GPD model with the set of parameters previously estimated. The dotted lines represent the curves obtained with the empirical excess returns. The plain and dotted curves nearly coincide, confirming both the appropriateness of the GPD approach and of its parameters.

While Lorenz curves provide a visual confirmation or negation of a model, it is possible to construct from these curves a quantitative index. This is the Gini coefficient which measures the ratio of the surface between the curve and the

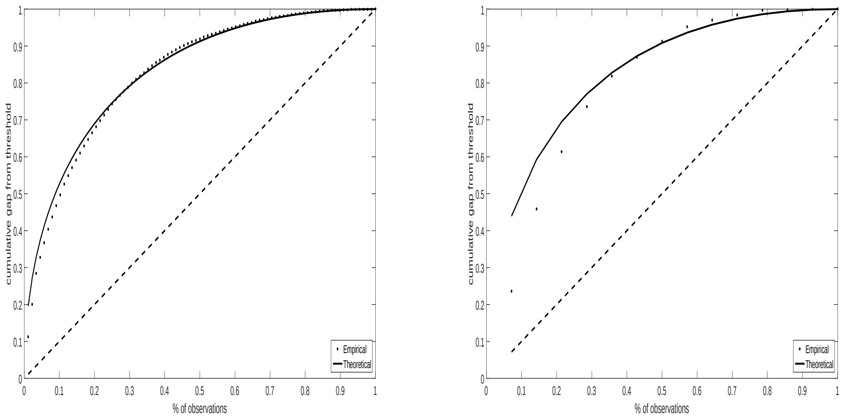

Figure 5 – Lorenz curves.

	G_t^n	G_e^n	G_t^p	G_e^p
Approach 1	72%	66%	66%	58%
Approach 2	67%	65%	65%	53%
Approach 3	66%	65%	65%	56%

Table 3 – Gini coefficients.

straight line to the surface of the half square. Table 3 gives the Gini coefficients associated with the historical bond data reconstructed using approaches 1, 2, and 3 of the previous section. In this table, G_e^n is the Gini coefficient for the empirical ('e') negative ('n') extremes, G_t^p is the Gini coefficient for the GPD theoretical ('t') positive ('p') extremes, and so on. We see that the greater level of validation of the GPD fit is achieved for the negative extremes that stem from approaches 2 and 3. These are the situations that matter most in practice.

POT graphs

The Peak Over Threshold, or POT, method is traditionally used to estimate u based on graphs representing the average empirical excess returns as a function of the threshold. POT graphs can also be used to confirm the validity of the GPD approach, and we use them for this purpose. If the GPD approach is valid, these graphs should represent points oscillating around a straight line. This is what is approximately observed in the POT graphs of Figure 6 that use the data reconstructed with the third approach of the previous section. This result confirms the validity of retaining a GPD approach. Note, though, that these graphs cannot be used for confirming our parameter estimates.

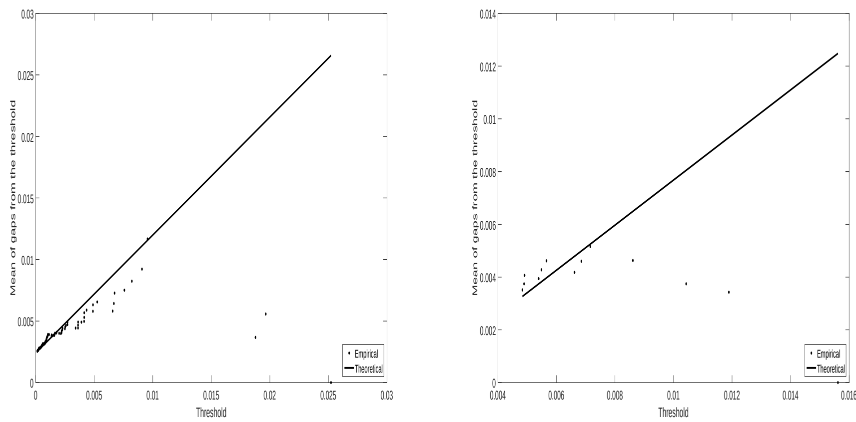

Figure 6 – POT graphs.

Kolmogorov-Smirnov test

Finally, we perform a Kolmogorov-Smirnov test. This test computes the maximum distance between a theoretical distribution function (here the GPD one) and the empirical distribution function. So, we compute

$$\theta = \max\left(\max\left(\frac{i}{n} - \mathrm{GPD}(x_i)\right), \max\left(\mathrm{GPD}(x_i) - \frac{i-1}{n}\right)\right),$$

where the x_i's are the empirical observations.

Table 4 gives the estimations of θ for both positive and negative extremes, the critical thresholds θ_c, and the p-values (all of them computed for a confidence

	θ^n	θ^n_c	p-value^n	θ^p	θ^p_c	p-value^p
Approach 1	0.11	0.29	0.96	0.20	0.34	0.52
Approach 2	0.07	0.14	0.79	0.21	0.39	0.65
Approach 3	0.07	0.14	0.80	0.17	0.35	0.74

Table 4 – Kolmogorov-Smirnov statistics and p-values.

level of 95%). The statistics are always inferior to the critical values and the p-values are always quite large in the table. From this test, we conclude that we cannot exclude the assumption that the GPD model is the right one. This is an additional confirmation that we can use the GPD for fitting bond data extreme values.

4 Solvency Capital Requirements

We can now compute the credit Solvency Capital Requirements associated with the portfolio of bonds shown in Table 6. A credit SCR measures the downward effect on the Net Asset Value of a credit scenario. This scenario should be the fiftieth worst one out of ten thousand scenarios: the goal is to protect the insurance company with a probability of 99.5%. For simplicity here, we restrict ourselves to a study of the impact of credit risk on the assets of the insurance company. The study of the effect of credit risk on the best estimate value of liabilities is left to a subsequent paper. Therefore, we adopt in this section the following simplified definition:

$$SCR = \frac{\mathcal{A}_0 - \mathcal{A}_{99.5\%}}{\mathcal{A}_0} \tag{12}$$

where $\mathcal{A}_0$ is the current value of the assets comprised of defaultable bonds and $\mathcal{A}_{99.5\%}$ the value of these bonds in the fiftieth worst catastrophic scenario out of ten thousand scenarios. We have divided the numerator by $\mathcal{A}_0$ for comparability reasons of the forthcoming results.

Insurers can compute their credit SCRs by either using a model such as that presented in this paper or by using a standard formula. The standard formula put forward in the European regulation (see the Commission Delegated Regulation (2015)) is as follows for a bond of sensitivity S:

$$SCR^{Std.Form.} = a + b\ (S - c) \tag{13}$$

where the parameters a, b, and c are chosen in the Tables 7, 8, and 9 depending on the sensitivity and rating of the bond. This SCR, similar to the definition of an SCR given in Eq. (12), is in fact a relative shock.

We show in Table 5 a comparison of the SCRs associated with the bonds of Table 6 and obtained using seven different methods. The first column gives the results obtained with the standard formula. The next columns give the results obtained with internal models. The second column of Table 5 computes the credit SCR by sorting the values of the bond portfolio reconstructed using approach 1, picking the 0.5% quantile, subtracting it from the current portfolio

	Standard Formula	1	1+GPD	2	2+GPD	3	3+GPD
Total Portfolio	5.5%	5.2%	6.1%	6.9%	7.9%	6.7%	7.2%
Sub Portfolio AAA	1.4%	0%	0%	0%	0%	0%	0%
Sub Portfolio AA	3.2%	6.4%	7.1%	11.5%	9.1%	9.1%	8.5%
Sub Portfolio A	6.1%	8.3%	8.9%	10.6%	11.4%	12.5%	10.6%
Sub Portfolio BBB	11.8%	7.1%	9%	8.9%	11.1%	8%	10.4%

Table 5 – Credit SCRs.

value, and renormalizing the difference by the current portfolio value. The SCRs for the subportfolios are computed as follows: the 0.5% quantile of the total bond portfolio is selected, a corresponding value of Π is deduced, and then the critical subportfolio value is computed using this value of Π. The third column of the table operates similarly on the data of approach 1 smoothed by the Generalized Pareto Distribution. Columns 4 to 7 are computed in a similar way as columns 2 and 3 but for approaches 2 and 3.

From Table 5, we make the following observations and interpretations. The internal models all give smaller SCRs than the standard formula for AAA[1] and BBB bonds. The opposite observation holds for AA and A bonds. This feature is a consequence of the nature of our database where we have selected completely risk-free AAA bonds and BBB bonds that have a higher recovery rate than AA and A bonds. For the total portfolio, we see that all internal models predict a higher SCR than the standard formula: this is a consequence of how this portfolio was built; for a portfolio with a higher proportion in the AAA and BBB bonds selected in our study, the results would be reversed.

From Table 5, we also observe that approaches 2 and 3, which are more accurate than approach 1, are more conservative (with or without GPD smoothing): they always predict larger SCRs. Then, if we compare approach 1 to approach 1 +GPD, approach 2 to approach 2+GPD, and approach 3 to approach 3+GPD, we see that it is not possible to conclude on the effect of smoothing: sometimes it increases and sometimes it decreases SCRs, without the effect being predictable at this stage.

Conclusion

This paper shows for the first time in the literature all the steps needed to produce a gross SCR associated with the credit risk of the assets of an insurance company. For most of these steps, new methods are proposed and tested. In the first section of the paper, new results on risk-premium adjustment factors are given. In the second section, innovative approaches for benchmarking credit portfolios are introduced. In the third section, a mixed GPD model for modeling credit-risky portfolios is introduced, estimated, and validated. In the fourth section, a case study compares the SCRs obtained using our approach to those obtained using the standard formula. One of the advantages of the framework introduced in this paper is that it takes into account the recovery risk of defaultable bonds, while the Solvency 2 standard formula does not.

A possible extension of our paper would concentrate on the computation of

[1] For AAA bonds, the models predict a null SCR because the recovery rate is 100% for all these bonds in our database.

the net credit SCR, namely of the SCR that also measures the impact of credit risk on the best estimate value of the insurance liabilities. To perform this computation, each primary simulation of the one-year credit risk should produce new values and ratings for the defaultable assets and should be supplemented by secondary simulations. This would allow us to obtain the value of economic capital - conditional on the primary shock simulated. Note that secondary simulations are also required in the absence of a primary shock in order to compute the time-0 value of economic capital. Other extensions of our framework could be done by introducing for instance stochastic recovery rates or by constructing a more refined model of rating transitions.

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Appendix A

Issuer	Rating	Recovery Rate	Maturity	Coupon	Dirty Price	Sensitivity
BEI	AAA	Full	11/10/2016	8%	115.49	1.7
FINANCEMENT FONCIER	AAA	Full	29/12/2021	5.62%	121.62	5.9
KFW	AAA	Full	21/01/2019	3.875%	119.08	3.7
GERMANY	AAA	Full	15/08/2023	2%	114.46	8
OAT	AA	Full	25/10/2019	3.75%	117.96	4.5
PROCTER	AA	44%	24/10/2017	5.125%	114.91	2.7
STATOIL	AA	40%	10/09/2025	2.875%	117.38	9.3
COMMONWEALTH	AA	40%	10/11/2016	4.25%	108.09	1.8
AIRBUS GP FIN.	A	55%	12/08/2016	4.625%	108.47	1.6
AIRBUS GROUP FIN.	A	55%	25/09/2018	5.5%	120.48	3.4
AIR LIQ.FIN	A	56%	15/10/2021	2.125%	110.02	6.3
CREDIT AGRICOLE	A	61%	22/12/2024	3%	101.01	8.4
PIRELLI INTER	BBB	64%	18/11/2019	1.75%	101.10	4.4
SEB	BBB	65%	03/06/2016	4.5%	107.71	1.4
VEOLIA	BBB	65%	24/05/2022	5.125%	131.83	6.3
URENCO FINANCE	BBB	60%	02/12/2024	2.375%	101.33	8.6

Table 6 – Bond dataset as of 31/12/2014

Appendix B

Sensitivity range	AAA	AA	A	BBB
0-5	0	0	0	0
5-10	4.5%	5.5%	7%	12.5%
10-15	7.2%	8.4%	10.5%	20%
15-20	9.7%	10.9%	13%	25%
20-	12.2%	13.4%	15.5%	30%

Table 7 – Parameter a of the standard formula.

Sensitivity range	AAA	AA	A	BBB
0-5	0.9%	1.1%	1.4%	2.5%
5-10	0.5%	0.6%	0.7%	1.5%
10-15	0.5%	0.5%	0.5%	1%
15-20	0.5%	0.5%	0.5%	1%
20-	0.5%	0.5%	0.5%	0.5%

Table 8 – Parameter b of the standard formula.

Sensitivity range	AAA-BBB
0-5	0
5-10	5
10-15	10
15-20	15
20-	20

Table 9 – Parameter c of the standard formula.